Problems in Attitude Control of Artificial "G" Space Stations with Mass Unbalance

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For a mass-unbalanced, axially-symmetrical space station consisting of a rotating artificial gravity section and a despun section, there exist coning motions with amplitudes that can be minimized by design such that the axial moment of inertia of the spinning section is as much as possible greater than the transverse moment of inertia of the space station. This implies that the despun section is much smaller than the spinning section and that the space station presents a low, flat silhouette when viewed from a direction normal to the spin axis. Also, crew compartments and compartments to and from which equipment is to be shifted should be situated as close as possible to the space station mass center. In addition, for very large satellites, it is found impractical to use control moment gyros to reduce the coning motion amplitudes to within allowable limits for many experiments which will be carried in future space stations. To alleviate the mass unbalance problem, a means of facilitating an active mass balance system is discussed, and a lighter, simpler, passive system involving a flexible interconnection between satellite sections is proposed for futher study.

Introduction

IN recent years several studies[1-4] have been conducted on the attitude motion of dual-spin satellites consisting of two bodies with relative motion restricted to rotations about a common axis fixed in both. Typically, these studies have been oriented toward small, unmanned satellites and have not considered several aspects pertinent to future large, long duration, manned space stations, such as attitude control by control moment gyros (CMG's) and significant mass unbalances due to shifts in locations of crew members and equipment.

NASA is now considering for the late 1970's or early 1980's a large manned space base which will provide artificial gravity and zero gravity environments simultaneously in separate sections.[5] Space Base, the method being considered for providing artificial gravity, is the spinning of the artificial g compartments so that objects within experience a radial acceleration directed toward the spin axis. However, these accelerations are imparted by forces of the satellite on the objects. The torques about the satellite mass center of the equal and opposite reaction forces of the objects on the satellite, if not balanced, far exceed the gravity gradient, aerodynamic, and other environmental torques and might be expected to have a correspondingly greater effect on attitude motions. Because future space stations will provide a significant degree of flexibility of movement for crew members and equipment, these torques inevitably will not be balanced.

Following is an analysis of the effect of mass center departures of the spinning section of a satellite from its spin axis upon the attitude motion and CMG requirements. Also, alternative methods for reducing the amplitudes of these motions are discussed.

System Description

The space station dynamical model to be considered is shown in Fig. 1 and consists of two sections attached on a common axis. The satellite is stabilized by control moment gyros providing three axis control on one of the sections. The other section is driven by a motor at a constant rate relative to the first about the common spin axis. Displacement of the mass center of the spinning section from the spin axis is represented by a mass particle attached to the spinning section.

System details and terminology are given as follows: The spinning section is termed body B and the other is termed body A. The spin axis intersects the mass centers of both bodies. The composite mass center is denoted by S^* and lies along principal axes of inertia of A and B, the corresponding principal moments of inertia being A_3 and B_3. For each body A and B, the principal moments of inertia about any line normal to the spin axis are equal and are denoted by A_1 and B_1, respectively. The rotation rate of B relative to A is ω, a constant. The mass particle is termed P with mass m and its position is designated by a distance l from S^* along the satellite spin axis and a distance r from the spin axis. The three CMG torque output axes are fixed with respect to A and lie along the spin axis (designated the 3 axis) and mutually perpendicular lines (designated the 1 and 2 axes) normal to the spin axis. If the orientation of this mutually right-handed orthogonal set of axes is given with respect to an inertially fixed set of axes by a 1,2,3 sequence of three axis Euler rotations ϕ_1, ϕ_2, and ϕ_3, then the CMG control torques are assumed given by

$$T_3^c = -K_{03}\phi_3 - K_{13}\dot{\phi}_3,\ T_i^c = -K_0\phi_i - K_1\dot{\phi}_i, \qquad i = 1,2 \quad (1)$$

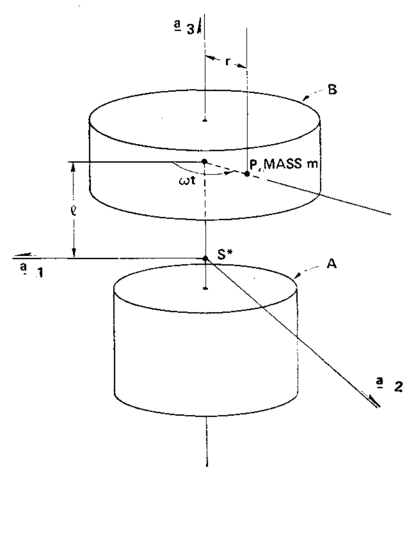

Fig. 1 Dual spin system.

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Equations of Motion

The exact dynamical equations of motion can be formulated by writing Euler's dynamical equations for the composite A-B body and the dynamical equations for the particle P separately. The equations for each contain the reaction force between A-B and P and elimination of that quantity results in the equations of motion for the whole system made up of A, B, and P. Gravity gradient and other environmental torques are neglected in the determination of the equations of motion because, for the rotation rate ω much greater than orbital rate, their magnitudes are very much smaller than the magnitude of torque associated with the mass particle P.

The exact dynamical equations of motion can be shown to be

$$\ddot{\phi}_1[J_1 + m(l^2 + r^2s^2\omega t)] - \ddot{\phi}_2 mr^2 s\omega tc\omega t - \ddot{\phi}_3 mrlc\omega t + \dot{\phi}_1[K_1 + 2\omega mr^2 s\omega tc\omega t] + \dot{\phi}_2\omega[B_3 + 2mr^2s^2\omega t] + \dot{\phi}_3 2\omega mrls\omega t + \phi_1 K_0 = -\omega^2 mrls\omega t + F_1(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \quad (2)$$

$$-\ddot{\phi}_1 mr^2 s\omega tc\omega t + \ddot{\phi}_2[J_1 + m(l^2 + r^2c^2\omega t)] - \ddot{\phi}_3 mrls\omega t - \dot{\phi}_1\omega[B_3 + 2mr^2c^2\omega t] + \dot{\phi}_2[K_1 - 2\omega mr^2 s\omega tc\omega t] - \dot{\phi}_3 2\omega rlc\omega t + \phi_2 K_0 = \omega^2 mrlc\omega t + F_2(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \quad (3)$$

$$-\ddot{\phi}_1 mrlc\omega t - \ddot{\phi}_2 mrls\omega t + \ddot{\phi}_3(A_3 + B_3 + mr^2) + \dot{\phi}_3 K_{13} + \phi_3 K_{03} = F_3(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t) \quad (4)$$

where $s\omega t = \sin\omega t$, $c\omega t = \cos\omega t$ and $F_j,(\phi_i,\dot{\phi}_i,\ddot{\phi}_i,\omega t)$, $i,j = 1,2,3$ represent nonlinear functions of $\phi_i,\dot{\phi}_i,\ddot{\phi}_i$ that are periodic in t of period $2\pi/\omega$ and which when expanded in a power series have all terms of the second power and higher in $\phi_i,\dot{\phi}_i,\ddot{\phi}_i$. J_1 is the principal moment of inertia of the composite A-B body at the composite mass center S^* in any direction normal to the spin axis.

For the above nonlinear differential equations with periodic coefficients, standard techniques of solution are not available. However, these can be expressed in a suitable form for which a method of successive approximations can be proved to yield steady state solutions to any desired accuracy.

Defining a new independent variable,

$$\tau = \omega/2\pi t = \bar{\omega}t \quad (5)$$

$$(d/dt)(\ ) = \bar{w}(d/d\tau)(\ ) = \bar{\omega}(\ )' \quad (6)$$

Equations (2–4) may be written

$$\phi_1''\{1 + \epsilon[d^2 + \tfrac{1}{2}(1 - c2p\tau)]\} - \phi_2''(\epsilon/2)sp\tau - \phi_3''\epsilon dcp\tau + \phi_1'(k_1 + \epsilon s2p\tau) + \phi_2'[b + \epsilon(1 - c2p\tau)] + \phi_3'2\epsilon dsp\tau + \phi_1 k_0 = -\epsilon dp^2 sp\tau + F_1(\phi_i,\phi_i',\phi_1'',p\tau)/J_1\bar{\omega}^2 \quad (7)$$

$$-\phi_1''(\epsilon/2)s2p\tau + \phi_2''\{1 + \epsilon[d^2 + \tfrac{1}{2}(1 + c2p\tau)]\} - \phi_3''\epsilon dsp\tau - \phi_1'[b + \epsilon(1 + c2p\tau)] + \phi_2'(k_1 - \epsilon s2p\tau) - \phi_3'2\epsilon dcp\tau + \phi_2 k_0 = \epsilon dp^2 cp\tau + F_2(\phi_i,\phi_i',\phi_i'',p\tau)/J_1\bar{\omega}^2 \quad (8)$$

$$-\phi_1''\epsilon decp\tau - \phi_2''\epsilon desp\tau + \phi_3''(1 + \epsilon e) + \phi_3'k_{13} + \phi_3 k_{03} = F_3(\phi_i,\phi_i',\phi_i'',p\tau)/J_1\bar{\omega}^2 \quad (9)$$

where

$$\epsilon = mr^2/J_1, \quad k_0 = K_0/\bar{w}^2J_1, \quad k_1 = K_1/\bar{\omega}J_1, \quad b = B_3/J_1, \quad k_{03} = K_{03}/\bar{w}^2(A_3 + B_3), \quad k_{13} = K_{13}/\bar{\omega}(A_3 + B_3), \ d = l/r, \quad e = J_1/(A_3 + B_3), \quad p = 2\pi \quad (10)$$

Solution of Eqs. (7–9) for ϕ_1'', ϕ_2'', ϕ_3'' and setting

$$X_i = \phi_i, \quad i = 1,2,3; \quad X_i = X_{i-3}', \quad i = 4,5,6 \quad (11)$$

gives equations of motion in a standard matrix form

$$X' = AX + f(X,\tau) + \epsilon g(\tau) \quad (12)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_0 & 0 & 0 & -k_1 & -b & 0 \\ 0 & -k_0 & 0 & b & -k_1 & 0 \\ 0 & 0 & -k_{03} & 0 & 0 & -k_{13} \end{bmatrix} \quad (13)$$

and $f(X,\tau)$ and $g(\tau)$ are periodic of period 1, with the average of $g(\tau)$ over a period equal to zero. It can also be shown that $f(X,t)$ consists of terms proportional to at least the first power in ϵ or nonlinear terms that approach zero as X^2 for X approaching zero, and there is a constant c, related to system parameters, such that†

$$|f(X,t) - f(Y,t)| \leq \epsilon c|X - Y| \quad (14)$$

for

$$|X| = \sum_{i=1}^{6} |X_i|, \ |Y| = \sum_{i=1}^{6} |Y_i|$$

sufficiently small. In addition Routhian analysis[6] can be used to show that the characteristic roots of matrix A have negative real parts for $k_0, k_1, k_{03}, k_{13} > 0$. Consequently, the results of Appendix A establish that for ϵ sufficiently small, the steady state solution to the equations of motion [Eq. (12) or Eqs. (7–9)] is periodic of period 1 in τ. Also, for sufficiently small ϵ, this solution is accurately represented by the first approximation of the method of successive approximations set forth in Appendix A.

The mass m is assumed to be very small in comparison to the total mass of bodies A and B. Then ϵ is a very small quantity and applying the method of successive approximation of Appendix A and neglecting terms of higher than the first power in ϵ yields

$$\phi_1 = \delta[-(1 + R_0)\sin p\tau + R_1\cos p\tau]/\rho^2 \quad (15)$$

$$\phi_2 = \delta[R_1\sin p\tau + (1 + R_0)\cos p\tau]/\rho^2, \ \phi_3 = 0 \quad (16)$$

where

$$\rho^2 = (1 + R_0)^2 + R_1^2 \quad (17)$$

where the dimensionless quantity δ is a measure of the mass unbalance and dimensionless quantities R_0 and R_1 are measures of the level of attitude control and

$$\delta = mrl/(B_3 - J_1), \quad R_0 = K_0/\omega^2(B_3 - J_1), \quad R_1 = K_1/\omega(B_3 - J_1) \quad (18)$$

Discussion of Solution

For small amplitude motions, the angle ψ, which the satellite spin axis makes with its nominal direction (that for which $\phi_1 = \phi_2 = 0$), is accurately given by

$$\psi = (\phi_1^2 + \phi_2^2)^{1/2} = \pm\delta/\rho \quad (19)$$

Since ψ is constant with respect to τ, the satellite spin axis once each period $T = 2\pi/\omega$ sec traces a right circular cone with half angle equal to ψ.

For the space station motion given by ϕ_1, ϕ_2, ϕ_3 of Eqs. (15–17), the minimum magnitude $\bar{H}$ of the total CMG spin angular momentum necessary to provide the control torques of

† The significance of Eq. (14) is developed in Appendix A.

Eq. (1) is determined in Appendix B as

$$\bar{H} = H_0(R_0^2 + R_1^2)^{1/2}/\rho \qquad (20)$$

where

$$H_0 = mrl\omega \qquad (21)$$

Also, the magnitude $\bar{T}$ of the CMG control torque can be shown to be

$$\bar{T} = (T_1^{c2} + T_2^{c2} + T_3^{c2})^{1/2} = \omega\bar{H} \qquad (22)$$

Now the values of ψ/δ and $\bar{H}/H_0$ can be plotted against R_0 for varying R_1. However, the nature of these curves depends upon whether $B_3 - J_1$ is greater than or less than zero. Defining

$$\delta^{\pm} = \pm\delta, \quad R_0^{\pm} = \pm R_0, \quad R_1^{\pm} = \pm R_1 \qquad (23)$$

where the positive sign is taken for $B_3 - J_1 > 0$, and the negative sign is taken for $B_3 - J_1 > 0$, Fig. 2 shows the variation of ψ/δ and $\bar{H}/H_0$ with respect to R_0^+ and R_1^+ and Fig. 3 shows the variation of ψ/δ and $\bar{H}/H_0$ with respect to R_0^- and R_1^-. These plots reveal that:

(1) The ψ/δ and $\bar{H}/H_0$ curves for $B_3 - J_1 > 0$ always lie below the corresponding curves for $B_3 - J_1 < 0$. Consequently, for minimizing the satellite coning angle and required CMG spin angular momentum, it is preferable for the space station design to be such that the moment of inertia of the spinning section about the spin axis is as much as possible greater than the moment of inertia of the space station about a direction normal to the spin axis.

2) For a space station with $B_3 - J_1 < 0$, values of K_0 in the region of $K_0 = \omega^2(J_1 - B_3)$ should be avoided.

3) For decreasing R_0^+, R_1^+ or R_0^-, R_1^- corresponding to decreasing CMG spin angular momentum, the space station coning angle approaches

$$\psi = \delta = |mrl/(B_3 - J_1)| \qquad (24)$$

This limiting value of $\psi = \delta$ shall hereafter be termed the light control coning angle. For increasing R_0^+, R_1^+ or R_0^-, R_1^-, the control moment gyros cannot have significant effect in decreasing the space station coning angle until the magnitude of

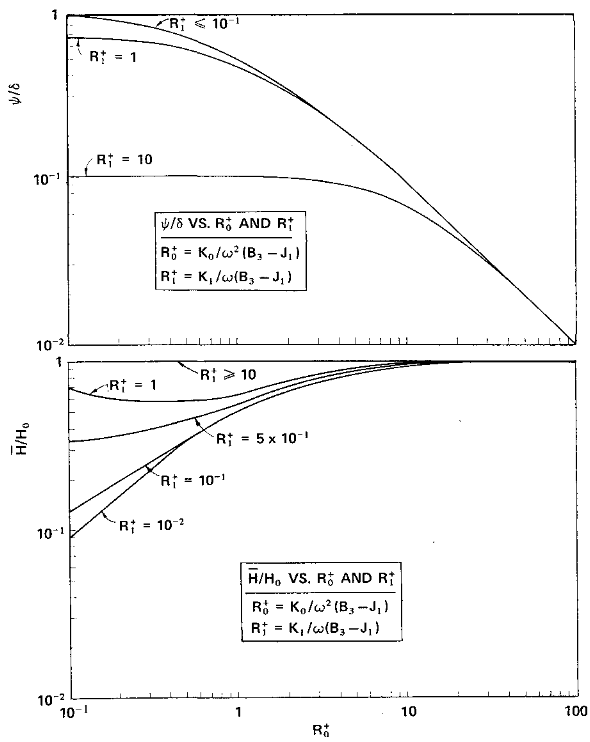

Fig. 2 Coning angle and minimum CMG momentum for $B_3 > J_1$.

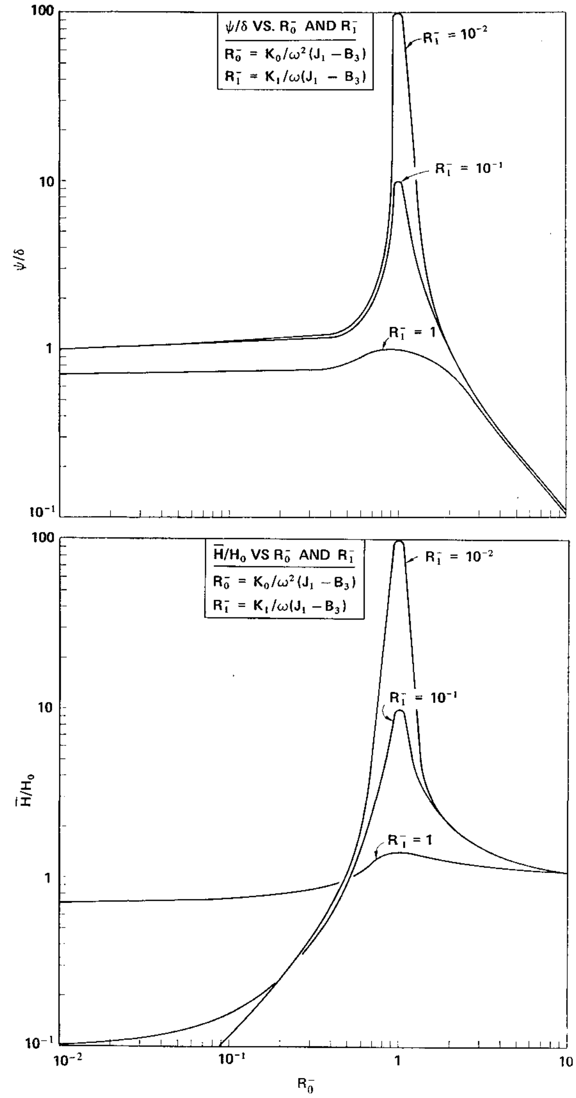

Fig. 3 Coning angle and minimum CMG momentum for $B_3 < J_1$.

their composite spin angular momentum is of the order of the limiting value

$$\bar{H} = H_0 = mrl\omega \qquad (25)$$

4) Both coning angle and minimum required CMG angular momentum are proportional to rl. Consequently, to minimize these quantities, crew compartments and compartments to and from which equipment is shifted should be located as close as possible to the satellite mass center.

Application to a Space Station Configuration

One configuration (Space Base) suggested for study by NASA[5a] is capable of supporting a crew of fifty and is comprised of a hub with a nonrotating section and a rotating section to which three arms are attached in a Y fashion and which is spinning so as to provide artificial gravity compartments at the extremities of one of the arms. This configuration has the worst unbalance characteristics for a supply spacecraft docked to the nonrotating section. For this case, the space base might be supposed to have the same (to within NASA's margin of error in their moment of inertia estimates) moment of inertia of $J_1 = 7 \times 10^8$ slug-ft² about any line normal to the spin axis and a spinning section axial moment of inertia of $B_3 = 9.5 \times 10^8$ slug-ft². Considering an unbalance mass of 300 slugs, equivalent to the mass of the Space Base elevator fully loaded and located at the artificial gravity compartment, then $r = 170$ ft, $l = 20$ ft and the light control coning angle is $\psi = \delta = 0.23°$.

A coning angle of 0.23° might well be unacceptable for several reasons. This amplitude of motion is outside the pointing accuracies required for some space station experiments,[7] especially those involving astronomical telescopes. Also, for this amplitude and $\omega = 4$ rpm, locations within the despun section and at distances greater than four feet from the space station mass center experience instantaneous accelerations greater than $10^{-4}\,g$. Some experiments, such as those involving crystal growth, require accelerations perhaps less than $10^{-5}g$.

Now, if for these or other reasons ψ is unacceptable, then, as declared in statement 3) previously, for a significant reduction, the CMGs must have a magnitude of total spin angular momentum of at least the order of H_0, $\bar{H} = H_0 = 425{,}000$ ft-lb-sec, a value very much beyond the capability of existing CMGs, optimistically 6000 ft-lb-sec for three CMGs.

Reduction of Amplitude of Motions

Mass Balancing System

It has been shown that, for very large artificial g space stations, CMGs offer little hope in reducing the amplitude of the angular oscillations induced by mass unbalance. Also, because of the large mass of fuel necessary, reaction jet attitude control systems appear impractical for long duration manned orbital missions. Consequently, special systems may well have to be employed. One such system which has been suggested[5b] involves providing counterbalance mass shifts by pumping of fluids or by movement of massive bodies.

The influencing factor in producing satellite coning results from the torque of the radial acceleration force of the mass unbalance particle. Components of this torque with respect to a β_1, β_2, β_3 axis system with origin at the satellite mass center, β_3 in the direction of the spin axis, and the axis system rotating about β_3 at rate ω are

$$M_1 = mr\omega^2 l \sin\theta, \quad M_2 = mr\omega^2 l \cos\theta \tag{26}$$

where θ is the angle between the β_1 axis and the radial line intersecting the unbalance mass.

Now if N movable counterbalance masses m_i can be shifted Δr_i, Δl_i, $\Delta\theta_i$ from their positions, represented by cylindrical coordinates r_i, l_i, θ_i, for which the mass center of the spinning section lies on the spin axis, then for the above unbalance torques M_1 and M_2, these shifts can be used to result in the two components of net torque about the satellite mass center being equal to zero.

$$\omega^2\left[mrl\sin\theta + \sum_{i=1}^{N}(m_i + \Delta m_i)(r_i + \Delta r_i)(l_i + \Delta l_i) \times \sin(\theta_i + \Delta\theta_i) - \sum_{i=1}^{N} m_i r_i l_i \sin\theta_i\right] = 0 \tag{27}$$

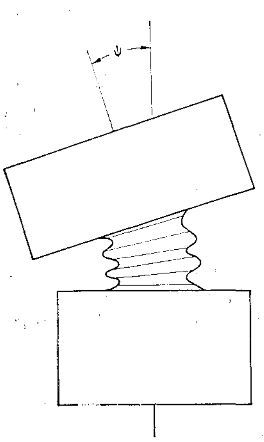

Fig. 4 Flexible connection between space station sections (ψ exaggerated).

$$\omega^2\left[mrl\cos\theta + \sum_{i=1}^{N}(m_i + \Delta m_i)(r_i + \Delta r_i)(l_i + \Delta l_i) \times \cos(\theta_i + \Delta\theta_i) - \sum_{i=1}^{N} m_i r_i l_i \cos\theta_i\right] = 0 \tag{28}$$

Since the quantities m_i, r_i, l_i, and θ_i are known quantities (indicating masses and locations of the counterbalance particles), then choosing at convenience all but two of the $4N$ quantities Δm_i (representing transfer of a fluid mass to or from a location), Δr_i, Δl_i, and $\Delta\theta_i$ (representing mass shifts), Eqs. (27) and (28) can be solved for the remaining two unknowns, provided the quantities m, r, l, and θ of the unbalance mass can first be found.

These are determined as follows: From Eqs. (15) and (16), the maximum amplitude L of ϕ_1 and ϕ_2 is given by

$$L = |mrl/(B_3 - J_1)\rho| \tag{29}$$

where R_0 and R_1 are given by Eq. (18). Then, mrl can be determined

$$mrl = |L(B_3 - J_1)\rho| \tag{30}$$

Also, values of $\tau = 0,1,2,\ldots$ indicate the instants at which the radial line from the spin axis to the unbalance mass is parallel to a unit vector in the direction of the first CMG output axis. Then, by Eqs. (16) and (29), at $\tau = 0,1,2,\ldots$, the ratio Q of the amplitude of ϕ_2 to its maximum amplitude is given by

$$Q = (1 + R_0)/\rho \tag{31}$$

Because the satellite oscillations ϕ_1 and ϕ_2 can be obtained with attitude sensors attached to the despun section of the satellite, then, since B_3, J_1, R_0, and R_1 are known quantities and the maximum amplitudes of the ϕ_1 and ϕ_2 oscillations can be measured from the sensor outputs, the quantity mrl can be determined from Eq. (30). Also, at the instant the measured ϕ_2 oscillation reaches the ratio Q of its maximum value, the radial line intersecting the unbalance mass is parallel to a known line and the angle θ of Eq. (26) can be determined. With this information, Eqs. (27) and (28) can be solved to give the mass shifts necessary to eliminate the satellite coning.

This method of mass balancing assumes that the spinning body B is axially symmetrical and that the spin axis lies along a principal axis of inertia of B. Consequently, some of the $4N$-2 quantities Δm_i, Δr_i, Δl_i, and $\Delta\theta_i$ which are prescribed must be chosen so as to insure that body B retains axial symmetry and that the principal axis of B remains along the spin axis. Also, steady state conditions have been assumed so that if the usual time intervals between mass shifts of crew and equipment are less than system decay time, or if velocities and accelerations of counterbalance masses are sufficiently large to produce potentially destabilizing effects, then transient effects must be considered.

Passive System

Since space station experiments requiring very small attitude motions would not likely be housed in the spinning artificial gravity section, it may be possible to bypass the coning problem by allowing the spinning section to experience the coning motion but providing an interconnection to the despun section that does not transmit significant torques in directions normal to the satellite spin axis. Then the amplitude of the motion performed by the despun section could be much smaller than that of the spinning section. One possible connection, shown in Fig. 4, might be a bellowslike tube which may or may not enclose a pressurized volume.

This means of alleviating satellite coning effects induced by mass unbalance of a spinning artificial gravity section offers the advantages over the active mass balance system of involving a passive system with greater simplicity and lower weight and, having apparently heretofore not been considered, should be investigated in the future.

Appendix A: Periodic Solutions of a Set of Differential Equations

A method of successive approximations has been set forth by Farnell, Langenhop, and Levinson[8], which provides arbitrarily good approximations to stable periodic solutions of systems of differential equations of the form

$$x' = Ax + f(x,\omega t) + \epsilon g(\omega t) \tag{A1}$$

where x is a vector with m components x_i, A is a constant $m \times m$ matrix with characteristic roots having negative real parts, the vector functions $f(x,t)$ and $g(t)$ are continuous and periodic in t of period 1 with the average of $g(t)$ over a period equal to zero, ϵ and ω are constants, and

$$|f(x,t) - f(y,t)| \leq \epsilon c|x - y| \tag{A2}$$

for small enough

$$|x| = \sum_{i=1}^{m} |x_i|, \ |y| = \sum_{i=1}^{m} |y_i|$$

uniformly in t and some constant c sufficiently small.

Then if ϵ is sufficiently small or ω is sufficiently large there exists a periodic solution $x^* = p(t)$ of period $1/\omega$. This is stable for $|x(t_0)|$ sufficiently small, and the steady state solution to Eq. (A1) as $t \rightarrow \infty$ is $x(t) = p(t)$ provided $\epsilon/(1 + \omega)$ is sufficiently small.

The statement (A2) is somewhat different from that given in Ref. 8, and the following is a correspondingly modified outline of the proof of the approximation method given there. If

$$x(t) = \epsilon \int_{-\infty}^{t} e^{A(t-\tau)}g(\omega\tau)d\tau + \int_{-\infty}^{t} e^{A(t-\tau)}f[x(\tau),\omega\tau]d\tau \tag{A3}$$

has a solution, then Eq. (A1) is satisfied by that solution. If a method of successive approximations is defined

$$x^{(0)}(t) = 0 \tag{A4}$$

$$x^{(n+1)}(t) = \epsilon \int_{-\infty}^{t} e^{A(t-\tau)}g(\omega\tau)d\tau + \int_{-\infty}^{t} e^{A(t-\tau)}f[x^{(n)}(\tau),\omega\tau]d\tau \tag{A5}$$

then

$$x^{(1)}(t) = \epsilon \int_{-\infty}^{t} e^{A(t-\tau)}g(\omega\tau)d\tau \tag{A6}$$

and for $|\int g(\omega\tau)d\tau|$ uniformly bounded, it can be shown that there is an N such that

$$|x^{(1)}(t)| \leq \max|x^{(1)}(t)| \leq \epsilon N/(1 + \omega) \tag{A7}$$

Now, since matrix A has characteristic roots with negative real parts, there exist k and $\sigma > 0$ such that

$$|e^{At}| \leq ke^{-\sigma t} \tag{A8}$$

and

$$|x^{(n+1)}(t) - x^{(n)}(t)| \leq k \int_{-\infty}^{t} e^{-\sigma(t-\tau)} \times |f[x^{(n)}(\tau),\omega\tau] - f[x^{(n-1)}(\tau),\omega\tau]|d\tau \tag{A9}$$

For $|x^{(n+1)}(\tau)|$ and $|x^{(n)}(\tau)|$ less than some λ, it has been given that

$$|f[x^{(n)}(\tau),\omega\tau] - f[x^{(n-1)}(\tau),\omega\tau]| \leq \epsilon c|x^{(n)} - x^{(n-1)}| \tag{A10}$$

so that if

$$M_n = \max|x^{(n)}(t) - x^{(n-1)}(t)| \tag{A11}$$

then, using Eqs. (A9) and (A10)

$$M_{n+1} \leq (\epsilon ck/\sigma)M_n \tag{A12}$$

and

$$|x^{(n)}| \leq \sum_{i=1}^{n} M_i \leq M_1\left[1 + \sum_{j=1}^{n-1} \left(\frac{\epsilon ck}{\sigma}\right)^j\right] \tag{A13}$$

For ϵ sufficiently small that $\epsilon ck/\sigma < 1$

$$|x^{(n)}| \leq M_1\left[1 + \frac{\epsilon ck/\sigma}{1 - (\epsilon ck/\sigma)}\right] \leq \max|x^{(1)}| \times \left[1 + \frac{\epsilon ck/\sigma}{1 - (\epsilon ck/\sigma)}\right] \tag{A14}$$

Hence, if $\epsilon/(1 + \omega)$ is sufficiently small that

$$[\epsilon N/(1 + \omega)][1 + (\epsilon ck/\sigma)/(1 - \epsilon ck/\sigma)] < \lambda \tag{A15}$$

these formulas are valid and for ϵ sufficiently small that $\epsilon ck/\sigma$ is a very small quantity, $x^{(1)}(t)$ is a good approximation to the steady state solution of Eq. (A1).

Appendix B: Angular Momentum Requirements for CMGs

The equations of motion for the CMG configuration may be written in terms of the configuration composite spin angular momentum ${}^c\mathbf{H}$, output torque $\mathbf{T}^c$, and angular velocity $\boldsymbol{\omega}^A$ of body A;

$${}^c\dot{\mathbf{H}} + \boldsymbol{\omega}^A \times {}^c\dot{\mathbf{H}} = -\mathbf{T}^c \tag{B1}$$

Or, written in terms of components associated with the directions of the three CMG output axes

$$\dot{H}_1 + u_2H_3 - u_3H_2 = K_1\ddot{\phi}_1 + K_0\dot{\phi}_1 \tag{B2}$$

$$\dot{H}_2 + u_3H_1 - u_1H_3 = K_1\ddot{\phi}_2 + K_0\dot{\phi}_2 \tag{B3}$$

$$\dot{H}_3 + u_1H_2 - u_2H_1 = K_{13}\ddot{\phi}_3 + K_{03}\dot{\phi}_3 \tag{B4}$$

Now the value of the u_i component of $\boldsymbol{\omega}^A$ is equal to $\dot{\phi}_i$, $i = 1,2,3$, to the first power in δ and substitution of $\dot{\phi}_i$ from Eqs. (15–17) for u_i in Eqs. (B2–B4) yields

$$\dot{H}_1 + \dot{\phi}_2H_3 = K_1\ddot{\phi}_1 + K_0\dot{\phi}_1 \tag{B5}$$

$$\dot{H}_2 - \dot{\phi}_1H_3 = K_1\ddot{\phi}_2 + K_0\dot{\phi}_2 \tag{B6}$$

$$\dot{H}_3 + \dot{\phi}_1H_2 - \dot{\phi}_2H_1 = 0 \tag{B7}$$

Multiplying Eq. (B5) by $\dot{\phi}_1$, Eq. (B6) by $\dot{\phi}_2$, and adding the result, it can be shown using Eqs. (15) and (16) that

$$\dot{H}_1\dot{\phi}_1 + \dot{H}_2\dot{\phi}_2 = \delta^2K_1(p_1^2 + p_2^2)\omega^2 \tag{B8}$$

where p_1 and p_2 are terms involving R_0 and R_1, which implies that H_1 and H_2 are proportional to δ and by Eq. (B7) H_3 is proportional to δ^2. Then, to the first power in δ,

$$\dot{H}_1 = K_1\ddot{\phi}_1 + K_0\dot{\phi}_1 \tag{B9}$$

$$\dot{H}_2 = K_1\ddot{\phi}_2 + K_0\dot{\phi}_2 \tag{B10}$$

$$\dot{H}_3 = 0 \tag{B11}$$

Integrating and considering the mean value of H_1, H_2, and H_3 to be zero in order to minimize $|{}^cH|$, then the amplitude $\overline{\mathbf{H}}$

of the ${}^{C}\dot{\mathbf{H}}$ vector is given to the first power in δ by

$$\bar{H} = (H_1^2 + H_2^2 + H_3^2)^{1/2} = H_0\{(R_0^2 + R_1^2)/[(1 + R_0)^2 + R_1^2]\}^{1/2} \quad \text{(B12)}$$

where R_0, R_1 are defined by Eqs.(18) and

$$H_0 = mrl\omega \quad \text{(B13)}$$

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